4 Variation

M11CH4

4.1. DIRECT VARIATION

Variation

If the values of two quantities depend on each other in such a way that a change in one quantity leads to a change in the other, then the two quantities are said to be in **variation**.

For example:

(i) As the money deposited in a bank increases, the interest also increases. Thus, the money deposited and the interest earned are in variation.

(ii) For a given job, as the number of workers increases, the time taken to complete the job decreases. Thus, the member of workers and the time for completion of the job are in variation.

Direct Variation

Consider the following table giving the number of articles (n) and their cost (c) in Liberian dollar (L$).

Number of articles (n)	2	6	10	14	20
Cost in L$ (c)	6	18	30	42	60

We observe that as the value of ***n* *increases***, the value of ***c* also increases** in such a way that the ratio $\frac{n}{c}$ (number of articles : Cost) remains constant, equal to $\frac{1}{3}$. We say n and c are in direct variation (or direct proportion).

Thus, two quantities x and y are said to be in **direct variation** or **direct proportion** if they increase (or decrease) together in such a manner that their ratio $\frac{x}{y}$ is same for all values of x and the corresponding values of y. The constant ratio $\frac{x}{y}$ is called the **constant** of **variation**.

Mathematically, if x and y are in direct variation, then we can write as:

$x \propto y$ and read as 'x varies directly as y' or 'x is directly proportional to y'.

Thus, $$x \propto y \quad \Rightarrow \quad \frac{x}{y} = k$$

where k is the constant of proportionality or variation.

$$\Rightarrow \qquad x = k\,y$$

If y_1, y_2 are values of y corresponding to the values x_1, x_2 of x respectively, then x and y are in direct variation implies

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} \quad \Rightarrow \quad \frac{x_1}{x_2} = \frac{y_1}{y_2}$$

$\Rightarrow$ Ratio of values of x = Ratio of values of y.

Notes:

- If two variables increase or decrease together, they need not be in direct variation always. For example, changes in weights and heights of individuals are not in direct variation.
- If y varies directly as x it implies as y increases x also increases such that their ratio remains constant.
- In order to solve problems under variation, following points should be kept in mind :
  (i) If x varies directly as the square of y, then $x \propto y^2 \Rightarrow x = ky^2$
  (ii) If x varies directly as the square root of y, then
  $x \propto \sqrt{y} \Rightarrow x = k\sqrt{y}$
  (iii) If x varies directly as the cube of y, then $x \propto y^3 \Rightarrow x = ky^3$
  (iv) If x varies directly as the cube root of y, then
  $x \propto \sqrt[3]{y} \Rightarrow x = k\sqrt[3]{y}$

Example 1: *In which of the following tables, x and y vary directly. Find the constant of variation if x and y are in direct variation.*

(*i*)

x	2	5	8.5	19
y	4	10	17	38

(*ii*)

x	4	6	8	15
y	12	12	20	40

Solution:

(i) Here $\frac{2}{4} = \frac{5}{10} = \frac{8.5}{17} = \frac{19}{38}$, each $= \frac{1}{2}$

$\Rightarrow$ The ratio of the corresponding values of x and y is constant and equal to $\frac{1}{2}$.

$\Rightarrow$ x and y are in direct variation

$\Rightarrow$ $x \propto y$

$\Rightarrow$ $x = ky$, where $k = \frac{1}{2}$

$\Rightarrow$ $x = \frac{1}{2} y$ or $y = 2x$

$\Rightarrow$ x is half of y or y is double of x.

(ii) Here, the ratio of the corresponding values of x and y is not same throughout.
Therefore, x and y are not in direct variation.

Example 2: *If x varies directly as y^2 and x = 36 when y = 3, find the value of x when y = 10.*

Solution: Given: x varies directly as y^2

$\Rightarrow$ $x \propto y^2 \Rightarrow x = ky^2$ ...(1)

x	36	?
y	3	10

Since $x = 36$ when $y = 3$, from (1), we have

$$36 = k \times 3^2 \Rightarrow k = \frac{36}{9} = 4$$

Therefore, $x = 4y^2$

When $y = 10$, $x = 4 \times 10^2 = 400.$

EXERCISE 4.1

1. In which of the following tables, x and y vary directly. Find the constant of variation if x and y are in direct variation.

(i)

x	2	5	9	15
y	6	15	27	45

(ii)

x	1	2	3	5	6
y	2	3	4	6	7

(iii)

x	2	5.5	9	14.5
y	8	22	36	58

2. If x and y are in direct variation, find the missing entries in the following tables:

(i)

x	3.5	...	...	17
y	7	10	18	...

(ii)

x	2	...	7	...
y	...	20	28	44

3. If $y \propto x$ and $y = 8$ when $x = 2$, find y when $x = 7.5$.

4. If $y \propto x^3$ and $y = 24$ when $x = 2$, find y when $x = 3$.

5. If $x \propto \sqrt{t}$ and $x = 6$ when $t = 9$, find

(i) x when $t = 25$ (ii) t when $x = 4$

6. y varies directly as x^2. If $y = 18$ when $x = 6$, find

(i) the constant of variation.

(ii) the value of y when $x = 5$

(iii) the value of x when $y = 8$.

Solving Problems Involving Direct Variations

Now we solve everyday life problems involving direct variation. We have already explained the following:

(i) $x \propto y \quad \Rightarrow \quad x = ky$

(ii) If x and y are in direct variation and

x	x_1	x_2
y	y_1	y_2

then $$\frac{x_1}{y_1} = \frac{x_2}{y_2} \quad \text{or} \quad \frac{x_1}{x_2} = \frac{y_1}{y_2}$$

Example 3: *A car travels 432 km on 48 litres of petrol. How far will it travel on 15 litres of petrol?*

Solution: Suppose the car travels x km on 15 litres of petrol. Then

Distance (in km)	432	x
Petrol (in litres)	48	15

We observe that **lesser** is the petrol consumed, **lesser** is the distance travelled. So, it is a case of direct variation.

Therefore, $$\frac{432}{48} = \frac{x}{15} \quad \Rightarrow \quad 9 = \frac{x}{15}$$

$$\Rightarrow \quad 9 \times 15 = x \quad \Rightarrow \quad x = 135$$

Hence, the car will travel 135 km on 15 litres of petrol.

Example 4: *A train is moving at a uniform speed of 80 km/hour.*

(*i*) *How far will it travel in 15 minutes?*

(*ii*) *What time will it take to cover a distance of 260 km?*

Solution: Suppose the train travels x km in 15 minutes and the time taken to cover 260 km in y minutes. Then

Distance travelled (in km)	80	x	260
Time taken (in minutes)	60	15	y

Since the speed is uniform, the distance travelled and the time taken are in direct variation. Both increase or decrease together. Therefore,

(i) $\frac{80}{60} = \frac{x}{15} \quad \Rightarrow \quad \frac{4}{3} \times 15 = x \quad \Rightarrow \quad x = 20$

Hence, the train will travel 20 km in 15 minutes.

(ii) $\frac{80}{60} = \frac{260}{y} \quad \Rightarrow \quad \frac{4}{3}\, y = 260 \quad \Rightarrow \quad y = 260 \times \frac{3}{4} \quad \Rightarrow \quad y = 195$

Hence, the train will take 195 minutes, i.e., 3 hours and 15 minutes to cover a distance of 260 km.

EXERCISE 4.2

1. A car travels 340 km in 5 hours with a uniform speed. In how many hours will it ravel 4080 km?

2. The cost of 12 identical books is L$ 720. Tabulate the cost of 3, 5 and 7 books of the same kind by using unitary method. Do the number of books and the cost vary directly as each other? If so, calculate the cost of 13 books by using the constant of variation.

3. If y varies directly as x and $y = 9$ when $x = 3$, find the value of y when $x = 4$.

4. r varies directly as the cube root of V. When $r = 5$, V = 125, find the value of r when V = 8.

5. y varies directly as the square of x. The table below shows the value of x for some selected values of y.

x	2	3	5	6
y	12	–	75	–

(i) Find the relation between y and x.

(ii) Find the value of the constant of proportionality.

(iii) Copy and complete the table

4.2. INVERSE VARIATION

We have seen that if two quantities increase or decrease together in the same ratio then they are in direct variation. Now, two quantities may change in such a manner that if one quantity increases, then the other quantity decreases and *vice versa*.

For example, suppose Emine wants to celebrate her birthday by distributing chocolates among her friends. She wants to spend L$ 300. How many chocolates will she get? It depends on the price of a chocolate.

Price of a chocolate (in L$)	6	10	12	20
Number of chocolates	50	30	25	15

We observe:

(i) As the price of a chocolate increases the number of chocolates that can be purchased with L$ 300 decreases.

(ii) The product of corresponding values of the two quantities is constant i.e., $6 \times 50 = 10 \times 30 = 12 \times 25 = 20 \times 15$, each = 300.

(iii) The ratio by which the price of a chocolate increases from 6 to 10 is 3 : 5 and the ratio by which the number of chocolates decreases is 50 : 30 = 5 : 3. Thus, the two ratios are inverses of each other.

If we represent the price of a chocolate by x and the number of chocolates by y, then as x increases y decreases and *vice versa*. The product xy remains constant. The ratio of any two values of x is the inverse of the ratio of corresponding values of y. We say 'x varies inversely with y' and 'y varies inversely with x'.

Thus, if two quantities x and y vary with each other in such a manner that the product xy remains constant, then we say that 'x and y vary **inversely**' or 'x and y are in **inverse variation**'. The product xy is called the constant of variation.

Mathematically, if x and y are in inverse variation, then we write we can as: $x \propto \dfrac{1}{y}$.

Thus, $$x \propto \frac{1}{y} \Rightarrow xy = k.$$

If y_1, y_2 are values of y corresponding to values x_1, x_2 of x respectively, then x and y are in inverse variation implies

$$x_1 y_1 = x_2 y_2 = k \quad \text{or} \quad \frac{x_1}{x_2} = \frac{y_2}{y_1}$$

$\Rightarrow$ The ratio of any two values of x = The inverse ratio of corresponding values of y.

Example 5: *In which of the following table x and y vary inversely:*

(*i*)

x	4	10	25	100
y	25	10	4	1

(*ii*)

x	8	12	16	20
y	9	6	5	3

(*iii*)

x	4	8	16	20
y	20	10	5	4

Solution:

(i) Here, $4 \times 25 = 10 \times 10 = 25 \times 4 = 100 \times 1$, each $= 100$

Since the product of the values of x and the corresponding values of y is constant, therefore, x and y vary inversely.

(ii) Here, $8 \times 9 = 12 \times 6 = 72$, $16 \times 5 = 80$, $20 \times 3 = 60$

Since the product of the values of x and the corresponding values of y is not the same throughout, therefore, x and y do not vary inversely.

(iii) Here, $4 \times 20 = 8 \times 10 = 16 \times 5 = 20 \times 4$, each $= 80$

Since the product of the values of x and the corresponding values of y is constant, therefore, x and y vary inversely.

Example 6: *If x and y are in inverse variation, fill in the blanks:*

x	*10*	*5*	...	*20*	...
y	*6*	...	*15*	...	*30*

Solution: Since x and y are in inverse variation, their product xy remains constant and equal to $10 \times 6 = 60$.

Therefore, first blank space must be filled by $\frac{60}{5} = 12$

Second blank space must be filled by $\frac{60}{15} = 4$

Third blank space must be filled by $\frac{60}{20} = 3$

Fourth blank space must be filled by $\frac{60}{30} = 2$

Example 7: *If x and y vary inversely and $x = 8$ when $y = 5$, find y when $x = 10$.*

Solution: Since x and y vary inversely, their product remains constant for all values of x and corresponding values of y. Then,

$$8 \times 5 = 10 \times y \Rightarrow 40 = 10\,y$$

$$\Rightarrow \quad y = \frac{40}{10} \Rightarrow y = 4$$

Example 8: *If x varies inversely as $\sqrt{y}$ and $x = \frac{5}{6}$ when $y = 36$. Find*

(*i*) *x when $y = 16$* (*ii*) *y when $x = 3$.*

Solution: Given : x varies inversely as $\sqrt{y}$

$$\Rightarrow \quad x \propto \frac{1}{\sqrt{y}} \Rightarrow x = \frac{k}{\sqrt{y}} \quad ...(1)$$

Given $x = \frac{5}{6}$ when $y = 36$, then from (1), we have

$$\frac{5}{6} = \frac{k}{\sqrt{36}} \Rightarrow \frac{5}{6} = \frac{k}{6} \Rightarrow k = 5$$

Putting value of k is (1), $\quad x = \frac{5}{\sqrt{y}} \quad ...(2)$

(i) When $y = 16$, from (1), we have

$$x = \frac{5}{\sqrt{16}} \Rightarrow x = \frac{5}{4}$$

(ii) When $x = 3$, from (1), we have

$$3 = \frac{5}{\sqrt{y}} \Rightarrow \sqrt{y} = \frac{5}{3} \Rightarrow y = \frac{25}{9}$$

Example 9: *If 12 workers can build a wall in 25 hours, how many workers will be required to do the same work in 20 hours?*

Solution: Let the required number of workers to build the wall in 20 hours be y.

Then

Number of hours	25	20
Number of workers	12	y

Clearly, more workers build the wall faster. Therefore, the number of hours and the number of workers are in inverse variation.

From the above table, we have

$$25 \times 12 = 20 \times y$$

$$\Rightarrow \quad y = \frac{25 \times 12}{20} = \frac{300}{20}$$

$$\Rightarrow \quad y = 15$$

Hence, 15 workers will build the wall in 20 hours.

EXERCISE 4.3

1. In which of the following tables x and y vary inversely:

(i)

x	8	6	24	2
y	9	12	3	36

(ii)

x	15	5	4	3
y	4	12	12	20

(iii)

x	10	5	25	8
y	10	20	4	12.5

2. If x and y vary inversely, fill in the blanks:

(i)

x	5	4	...	8
y	8	...	20	...

(ii)

x	12	6	...	3
y	...	16	4	...

3. If y varies inversely as x and $y = 8$ when $x = 3$, find

(i) the constant of variation

(ii) the relation between y and x.

(iii) the value of x when $y = 12$.

4. If y varies inversely as the square of x and $y = 100$ when $x = 3$, find

(i) the relation between x and y.

(ii) the value of x when $y = 25$.

(iii) the value of y when $x = 15$

5. If y varies inversely as x^3 and $y = 3$ when $x = 2$, find

(i) the constant of variation.

(ii) the value of y when $x = 4$.

(iii) the value of x when $y = 81$.

6. If P is inversely proportional to Q^2 and $Q = 5$ when $P = 2$, find

(i) the constant of variation

(ii) the positive value of Q when $P = \frac{1}{2}$.

7. P varies inversely as Q. The table below shows the value of Q for some selected values of P.

P	6	8	9	12
Q	24	...	...	12

(i) Find the relation between P and Q.

(ii) Find the value of the constant of proportionality.

(iii) Copy and complete the table.

8. If y is inversely proportional to $(x + 2)$ and $y = 48$ when $x = 10$, find x when $y = 30$.

9. If 52 men can do a piece of work in 35 days, then in how many days will 28 men do it?

10. A garrison of 400 men had food for 40 days. After 10 days, 200 more men joined them. How long will the food last now? (Assume that the amount of food taken by each man is almost the same).

4.3. JOINT VARIATION

If the value of a quantity depends on two or more other quantities in such a way that a change in one quantity leads to the change in other quantities, then, these quantities are said to be in **joint variation**.

For example:

(i) The electrical resistance R Ω of a wire varies directly as the length L cm and inversely as the square root of the diameter *d* cm i.e.;

$$R \propto \frac{L}{\sqrt{d}} \quad \text{or} \quad R = \frac{kL}{\sqrt{d}},$$

where k is the constant of proportionality.

(ii) The height h of a cylinder varies directly as the mass m and inversely as the square of its radius. i.e.,

$$h \propto \frac{m}{r^2} \quad \text{or} \quad h = \frac{km}{r^2},$$

where k is the constant of proportionality.

Example 10: *The volume (V) of a cylinder varies directly as product of the square of its radius (r) and height (h). Given that h = 6 m and r = 6 cm. Find the volume in cm^3 of the cylinder if the constant of proportionality is π.*

Solution: As per given,

$$V \propto r^2 h \quad \Rightarrow \quad V = kr^2 h,$$

where k is the constant of proportionality.

Now, $h = 6 \text{ m} = 6 \times 100 \text{ cm} = 600 \text{ cm}$, $r = 6$ cm, $k = \pi = \dfrac{22}{7}$

∴ The required volume is given as:

$$V = \pi r^2 h = \frac{22}{7} \times 6^2 \times 600$$

$$= 67885.71429 \text{ cm}^3 \cong 67886 \text{ cm}^3.$$

Example 11: *The electrical resistance R of a wire varies inversely as the cube root of the diameter d and directly as the length L.*

Find the value of d correct to 3 significant figures when L = 25 cm, R = 0.18 Ω and k = 1.25 × 10^{-3}.

Solution: As per given,

$$R \propto \frac{L}{d^{1/3}} \quad \Rightarrow \quad R = \frac{kL}{d^{1/3}} \qquad \text{...(1)}$$

where k is the constant of proportionality.

Cubing both sides, (1) gives

$$R^3 = \frac{k^3L^3}{d} \quad \Rightarrow \quad L = \frac{k^3L^3}{R^3} \qquad \text{...(2)}$$

Substituting L = 25 cm, R = 0.18 Ω, $k = 1.25 \times 10^{-3}$ in (2) to get

$$d = \frac{(1.25 \times 10^{-3})^3 \times (25)^3}{(0.18)^3}$$

$$= 5.232780886 \times 10^{-3} \text{ cm}$$

$$\cong 0.0523 \text{ cm.}$$

Example 12: *An electromagnetic force F between two objects varies directly as the product of their masses M and m and inversely as the square of the distance r between them. Given that: F = 50 N, M = 25 kg, m = 12 kg, r = 2 m. Find the value of r (in cm) when F = 30 N, M = 7.5 kg, m = 2.5 kg.*

Solution: As per given, $F \propto \frac{Mm}{r^2} \quad \Rightarrow \quad F = \frac{kMm}{r^2}$ ...(1)

where k is the constant of proportionality.

Using F = 50 N, M = 25 kg, m = 12 kg, r = 2 m, in (1) to get k as:

$$\therefore \quad 50 = \frac{k \times 25 \times 12}{4} \quad \Rightarrow \quad k = \frac{200}{300} = \frac{2}{3}$$

$$\therefore \quad \text{(1) gives} \quad F = \frac{2}{3} \times \frac{Mm}{r^2} \qquad \text{...(2)}$$

When F = 30 N, M = 7.5 kg, m = 2.5 kg, (2) gives

$$30 = \frac{2}{3} \times \frac{7.5 \times 2.5}{r^2} = \frac{37.5}{3r^2}$$

$$\Rightarrow \quad r^2 = \frac{37.5}{3 \times 30} = \frac{375}{900} = \frac{15}{36} = \frac{5}{12}$$

$$\Rightarrow \qquad r = 0.645497224 \text{ m}$$
$$\cong 0.6455 \text{ m}$$
$$\cong 64.55 \text{ cm}$$

Example 13: *A quantity x varies directly as y and inversely as the square of z. Some selected values of x, y and z are shown in the following table:*

(*i*) *Find the constant of proportionality.*

(*ii*) *Find a formula between x, y and z.*

(*iii*) *Complete the table.*

x	12	...	3
y	8	8	...
z	2	2	4

Solution: As per given, $x \propto \dfrac{y}{z^2} \Rightarrow x = \dfrac{ky}{z^2}$, ...(1)

where k is the constant of proportionality.

(*i*) Given that when $x = 12$, $y = 8$, $z = 2$, then from (1), we have

$$12 = \frac{8k}{4} \quad \Rightarrow \quad k = 6$$

(*ii*) Substituting $k = 6$ in (1), we get

$$x = \frac{6y}{z^2} \qquad \text{...(2)}$$

which is the required formula.

(*iii*) When $y = 8$, $z = 2$, (2) gives

$$x = \frac{6 \times 8}{4} = 12$$

When $x = 3$, $z = 4$, (2) gives

$$3 = \frac{6y}{10} \Rightarrow 6y = 48, \quad y = \frac{48}{y} = 8$$

$\therefore$ The complete table is shown below:

x	12	12	3
y	8	8	8
z	2	2	4

EXERCISE 4.4

1. A quantity m varies directly as n and inversely as the square of p. Given that $m = 3$ when $n = 2$ and $p = 1$. Write down an equation connecting m, n and p.

2. If x varies directly as y and inversely as z, then write down an equation connecting x, y and z.

Given that $x = 8$ when $y = 5$ and $z = 3$. Find y when $x = 12$ and $z = 8$.

3. The electrical resistance R ohms of a wire varies directly as the length L cm and inversely as the square root of the diameter d cm. Find:

(*i*) an expression of d in terms of L, R and the constant of proportionality k.

(*ii*) the value of d correct to 2 decimal places when L = 15 cm, R = 0.13 ohms and $k = 1.25 \times 10^{-3}$.

4. Three quantities P, Q and R are connected so that P varies directly as R and inversely as the square root of Q. If P = 6 when R = 12 and Q = 25, find

(*i*) an expression for P in terms of Q and R

(*ii*) the value of Q when P = 30 and R = 9

5. The length L cm of a wire that can be coiled on a cylinder varies as the radius r cm of the cylinder and inversely as the square of the diameter d cm, of the wire. On a cylinder of radius 5.5 cm, a wire 68.5 m in length and 0.5 cm thick can be coiled. Find:

(*i*) The relation between the length, and diameter of wire and the radius of the cylinder

(*ii*) The length of wire, correct to three significant figures, 0.8 cm thick, that can be coiled on a cylinder of radius 20 cm.

6. The height h of a cylinder X of a given material varies directly as the mass m and inversely as the square of the radius r. The height of X is 12 cm. If is required to make another cylinder Y of the same material. If the mass of Y is $\frac{1}{4}$ the mass of X and its radius 0.4 that of X, calculate, correct to three significant figures, the height of Y.

7. F varies directly as the product of M and m and inversely as the square of d. Given that F = 20 when M = 2.5, $m = 10$ and $d = 5$. Find:

(*i*) an expression of F in terms of M, m and d.

(*ii*) the value of d, when F = 30, M = 7.5 and m = 4.

8. P varies directly as R and inversely as the square of Q, where, P, R and Q are positive integers. The table below shows that value of P for some selected values of R and Q.

P	6	...	3
R	1	8	16
Q	1	2	...

(*i*) Find the relation between P, R and Q.

(*ii*) Find the value of the constant of proportionality.

(*iii*) Copy and complete the table.

9. In the table, $W \propto \frac{Q}{R^2}$, where W, R and Q are positive integers. Solve for w_2 and r_3.

W	R	Q
3	4	4
w_2	1	2
8	r_3	6

4.4. PARTIAL VARIATION

In partial variation, a given quantity (variable) is related to two or more other quantities (variables) added together. This type of variation is sum of two or more variations.

For example, a quantity x is a partly constant, and partly varies directly or inversely as y. To represent different partial variation, we adopt the following rules. If k_1 is constant and k be the constant of proportionality, then,

(i) If x is partly a constant and partly varies directly as y, then

$$x = k_1 + ky.$$

(ii) If x is partly a constant and partly varies inversely as y, then

$$x = k_1 + \frac{k}{y}.$$

(iii) If x is partly a constant and varies jointly as y and z, then

$$x = k_1 + kyz.$$

(iv) If x varies partly as y and inversely as the square of z, then

$$x = k_1 y + \frac{k_2}{z^2}$$

where k_1 and k_2 are constants of proportionality.

Note: For partial variation, we always have two constants which can be determined by solving two simultaneous equations (formed by given conditions in the question).

Example 14: *A quantity is partly a constant and partly varies as the square of another quantity by the relation* $y = k_1 + kx^2$*; k and* k_1 *are constants. Given that* $y = 40$ *when* $x = 1$ *and* $y = 13$ *when* $x = 2$*. Find the value of y when* $x = \frac{7}{3}$.

Solution: Given relation is

$$y = k_1 + kx^2 \quad ...(1)$$

When $x = 1$, $y = 40$, then (1) gives $40 = k_1 + k$ ...(2)

When $x = 2$, $y = 13$, then (1) gives $13 = k_1 + 4k$ ...(3)

Subtracting (2) and (3) to get value of k as :

$$27 = -3k \Rightarrow k = -9$$

Again from (2), $40 = k_1 - 9 \Rightarrow k_1 = 49$

$\therefore$ (1) gives $y = 49 - 9x^2$ ...(4)

When $x = \frac{7}{3}$, (4) gives

$$y = 49 - 9 \times \frac{49}{9} = 0.$$

Example 15: *The cost of manufacturing a lead pencil is partly a constant and partly varies inversely with the number of lead pencils manufactured per hour. If 1000 lead pencils are manufactured per hour, the cost is L$ 3 per lead pencil. If 2000 lead pencils are manufactured per hour, the cost is reduced to L$ 2.5 per lead pencil.*

(i) Find the cost of each lead pencil if manufacturing 5000 lead pencils per hour.

(ii) How many lead pencils can be manufactured at a cost of L$ 4 per lead pencil?

Solution: Let L$ x be the cost of manufacturing a lead pencil and y be the number of lead pencils manufactured per hour. Then, as per given

$$x = k_1 + \frac{k}{1000}, \quad \text{...(1)}$$

where k_1 is constant and k_1 is the constant of proportionality.

Given $x = 3$, $y = 1000$, then (1) gives

$$3 = k_1 + \frac{k}{1000} \quad \text{...(2)}$$

Given $x = 2.5$, $y = 2000$, then (1) gives

$$2.5 = k_1 + \frac{k}{2000} \quad \text{...(3)}$$

Subtracting (3) from (2) to get value of k as :

$$0.5 = k\left(\frac{1}{1000} - \frac{1}{2000}\right) = \frac{k}{2000}$$

$$\Rightarrow \quad k = 2000 \times 0.5 = 1000$$

Also (2) gives, $$3 = k_1 + \frac{1000}{1000} = k_1 + 1 \quad \Rightarrow \quad k_1 = 2$$

Substituting the values of k and k_1 in (1) to get

$$x = 2 + \frac{1000}{y} \quad \text{...(4)}$$

(i) If $y = 5000$, then (4) gives

$$x = 2 + \frac{1000}{5000} = 2 + 0.2 = \text{L\$ } 2.2$$

(ii) When $x = 4$, then (4) gives

$$4 = 2 + \frac{1000}{y} \quad \Rightarrow \quad 2 = \frac{1000}{y}$$

$$\Rightarrow \quad y = \frac{1000}{2} = 500$$

Hence, 500 lead pencils can be manufactured at the cost of L$ 4 per lead pencil.

Example 16: *The water bill in L$ x of a factory in Liberia is partly a constant and partly varies directly as the cube of the volume v litres of water consumed.*

Given that when v = 500 litres, x = L$ 15.75 and when v = 600 litres, x = L$ 14.25. Find the water bill of the factory when it has consumed 200 litres of water.

Solution: As per given, $x = k_1 + kv^3$, ...(1)

where k_1 is a constant and k is the constant of proportionality and are to be determined.

Given that when $v = 500$, $x = 15.75$, (1) gives

$$15.75 = k_1 + 125 \times 10^6\, k \qquad ...(2)$$

When $v = 600$, $x = 14.25$, (1) gives

$$14.25 = k_1 + 216 \times 10^6\, k \qquad ...(3)$$

Subtracting (3) from (2) to get value of k as :

$$1.5 = (125 - 216) \times 10^6\, k$$
$$= -91 \times 10^6\, k$$

$$\Rightarrow \qquad k = -\frac{1.5}{91 \times 10^6} = \frac{-15}{91 \times 10^7}$$

$$= -0.164835164 \times 10^{-7}$$

Substituting the value of k in (2), we get

$$15.75 = k_1 + (125 \times 10^6) \times (-0.164835164 \times 10^{-7})$$
$$= k_1 - 20.6043955 \times 10^{-1}$$
$$= k_1 - 2.06043955$$
$$\Rightarrow \qquad k_1 = 17.81043955$$

Substituting the values of k and k_1, (1) gives

$$x = 17.81043955 - 0.164835164 \times 10^{-7}\, v^3 \qquad ...(4)$$

We are required to find x when v = 200 litres.

When $v = 200$, (4) gives

$$x = 17.81043955 - (0.164835164) \times 10^{-7} \times (200)^3$$
$$= 17.81043955 - (0.164835164) \times 8 \times 10^{-1}$$
$$= 17.81043955 - 0.131868731$$
$$= 17.67857142$$
$$\cong 17.68$$

$\therefore$ The cost of consuming 200 litres of water is L$ 17.68.

Example 17: *The cost of running a private school in Monrovia is partly a constant and partly varies directly as the number of pupils. If the school has 100 pupils, the running cost is US$ 19,475 per year and with 75 pupils, the cost is US$ 17,575 per year.*

(i) *What will be the running cost of the school per year if it has only 45 pupils?*

(ii) *If the fee per pupil is US$ 1,400 per year, find the least number of pupils required for which the school can run without a loss.*

Solution: Let the cost of running the private school is US$ x and y be the number of pupils.

As per given, $$x = k_1 + k\,y \qquad ...(1)$$

where k_1 and k are constants and are to be determined.

Given, when $y = 100$, $x = 19475$, then (1) gives

$$19475 = k_1 + 100\,k \qquad ...(2)$$

When $y = 75$, $x = 17575$, then (1) gives,

$$17575 = k_1 + 75\,k \qquad ...(3)$$

Subtracting (3) from (2) to get the value of k as:

$$1900 = 25k \quad \Rightarrow \quad k = 76$$

Substituting the value of $k = 76$ in (2), we get

$$19475 = k_1 + 100 \times 76 = k_1 + 7600$$

$$\Rightarrow \qquad k_1 = 19475 - 7600 = 11875$$

Substituting the values of k and k_1 in (1), it becomes

$$x = 11875 + 76\,y \qquad ...(4)$$

(*i*) We are required to find the value of x when $y = 45$.

$\therefore$ When $y = 45$, (4) gives

$$x = 11875 + 76 \times 45$$
$$= 11875 + 3420 = 15295$$

$\therefore$ The required cost of running the school with 45 pupils is US$ 15295.

(*ii*) Let y be the least number of required pupils. If the fee per pupil is US$ 1,400, then, cost of running the school is $x = 1400\,y$

$\therefore$ (4) gives $\qquad 1400\,y = 11875 + 76\,y$

$\Rightarrow \qquad 1324\,y = 11875$

$$\Rightarrow \qquad y = \frac{11875}{1324} = 8.969033233$$

$$\cong 9$$

$\therefore$ The required number of pupils for running the school for no loss is 9.

EXERCISE 4.5

1. A quantity y is partly a constant and partly varies as the square of x. If $y = 40$ when $x = 1$ and $y = 13$ when $= 22$, find the value of y when $x = 3$.

2. The cost of producing a radio component is partly a constant and partly varies inversely with the number of components produced per day. If 100 components are produced per day, the cost is $6.00 per component. If 200 components, the cost is reduced to $4.50 per component.

(i) What is the cost of each component if 500 are produced per day?

(ii) How many components can be produced per day at a cost of $7.00 per component?

3. The cost (c) of producing a motor car in a certain factory is partly constant and partly varies inversely as the number (n) of cars produced per day. The cost of producing 4 cars per day is $1,600 and that of producing 5 cars per day is $1,420. Find the relation between c and n. Find the number of cars produced per day necessary to bring the cost down to $1,000 per day.

4. The resistance, R to the motion of a car is partly constant and partly varies as the square of the speed, V. When the car is moving at 30 km/h, the resistance is 630 N and at 50 km/h, the resistance is 950 N. Find

(i) an expression for R in terms of V

(ii) the resistance at 80 km/h.

5. In a community, water bill, R (in L$), is partly constant and partly varies as the cube of the volume, v litres of water consumed. When $v = 50$ litres, R = L$ 4.25 and when $v = 60$ litres, R = L$ 5.16. Find an expression for R in terms of v.

6. The cost of maintaining a shool is partly a constant and partly varies as the number of pupils. With 50 pupils, the cost is \$15,705.00 and with 40 pupils, the cost is \$13,305.00.

(i) What is the cost when there are 44 pupils?

(ii) If the fee per pupil is \$360.00, what is the least number of pupils for which the school can run without a loss?

MULTIPLE CHOICE QUESTIONS

1. If decrease in one quantity leads to a corresponding decrease in other quantity, then, this is a

(a) direct variation (b) indirect variation

(c) joint variation (d) none of these

2. If a quantity x is directly proportional to y, then

(a) y is directly proportional to x

(b) y is inversely proportional to x

(c) y is jointly proportional to x

(d) none of these

3. If x is directly proportional to y and y is inversely proportional to z. Then what is the proportionality between x and z?

(a) x is directly proportional to z (b) x is inversely proportional to z

(c) x is partially proportional to z (d) x is jointly proportional to z

4. If x and y are directly proportional and when $x = 7$, $y = 21$, which of the following is not possible pair of corresponding values of x and y?

(a) 6 and 18 (b) 13 and 39

(c) 30 and 10 (d) 17 and 51

5. If $y \propto \sqrt{x}$ and $x = 16$ when $y = 2$, then the value of constant of variation k is

(a) $k = \frac{1}{2}$ (b) $k = \frac{1}{3}$

(c) $k = 1$ (d) $k = 2$

6. In Q. 5 above, the value of x when $y = 24$ is

(a) 288 (b) 576

(c) 1152 (d) 2304

7. If $y \propto x^2$ and if $y = 18$ when $x = 6$, then, the value of y when $x = 8$ is

(a) 12 (b) 24

(c) 32 (d) 36

8. If $x \propto \frac{1}{y}$ and $x = 5$ when $y = 4$. Then, the constant of variation is

(a) 20 (b) 10

(c) 5 (d) none of these

9. The cost of a piece of cloth is L$ 1940 per 40 m. For L$ 727.5, one can purchase x metres of cloth where

(a) $x = 10$ m (b) $x = 12$ m

(c) $x = 13$ m (d) none of these

10. Let 11 men can dig $6\frac{3}{4}$ metre long trench in one day. To dig 27 metre long trench, the number of men to be employed is

(a) 40 (b) 44

(c) 50 (d) 54

11. If 52 men can do a piece of work in 35 days. The same work can be done in 65 days with

(a) 28 men (b) 32 men

(c) 30 men (d) 25 men

12. If $F \propto Mm$ and $F \propto \frac{1}{d^2}$. Given that $F = 20$ when $M = 2.5$, $m = 10$, $d = 5$. Then, the value of F when $M = 7.5$, $m = 4$, $d = 4.47$ is

(a) 32 (b) 30

(c) 30.47 (d) 32.47

13. If $y \propto$ partly as the square of x and y is partly a constant. Given that $y = 40$ when $x = 1$ and $y = 13$ when $x = 2$. The positive value of x when $y = -32$ is

(a) 2 (b) 3

(c) 4 (d) 6

14. A quantity y is partly a constant and partly varies inversely as square of x. The relationship between x and y is

(a) $y = 3 + \frac{4}{x^2}$ (b) $y = 3 - \frac{4}{x^2}$

(c) $y = 3 + \frac{8}{x^2}$ (d) $y = 3 - \frac{8}{x^2}$